$$\gamma = \left(\frac{\Omega g}{2\pi} k^2 c^2\right)^{1/3} = \frac{(\alpha S)^{2/3}}{\tau_a}, \tag{20}$$

where $\Omega = 1/\sigma a$ is the resistance of the current layer per unit length. If we insert the value $\alpha = S^{-1/4}$ into (20) – this being the value at which the maximum increment is obtained, then we obtain the correct relation $\tau_a \gamma_{\max} \sim S^{1/2}$ [5]. There is also quantitative agreement with the results of numerical calculations in the longwave region $\alpha < S^{-1/4}$ (Fig. 2, 1 – (20), 2 –calculation from [5]).

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EVOLUTION OF AN INTENSE SPHERICAL SHOCK WAVE IN AN INHOMOGENEOUS ATMOSPHERE

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After the publication of [1] special interest arose in the propagation of intense spherical [2-6] and planar [3, 7, 8] shock waves in an inhomogeneous atmosphere. The "geometric dynamics" of intense shock waves has been developed [9, 10], and a "characteristic rule" has been proposed for such waves [10, 11]. The technique of [9-11] has been applied successfully to planar waves [10].

Relying on the concepts of [9-11] the present study will examine evolution of an intense spherical shock wave in an inhomogeneous atmosphere. The accuracy of the approximate analytical expressions obtained proves to be higher than the analogous results of [1-5].

The undisturbed state of atmospheric density and pressure are characterized by the expression $\rho_0(z)/\rho_0(0) = p_0(z)/p_0(0) = \exp(-z/H)$ (where H is the height of the "homogeneous" atmosphere). We will describe the medium by a system of gas dynamics equations

$$\begin{gathered}\partial\rho/\partial t + \operatorname{div}(\rho\mathbf{v}) = Q_1,\ d\mathbf{v}/dt = -(1/\rho)\nabla p + \mathbf{g} = \mathbf{Q}_2,\\ dp/dt - a^2 d\rho/dt = Q_3,\end{gathered} \tag{1}$$

where $\mathbf{v}$, $\mathbf{g}$, a are gas velocity, the acceleration of gravity, and the speed of sound; $Q_1(t, \mathbf{r}) = Q_{01}(t)\delta(\mathbf{r})$; $\mathbf{Q}_2(t, \mathbf{r}) = \mathbf{Q}_{02}(t)\delta(\mathbf{r}) || \mathbf{z}$; $Q_3(t, \mathbf{r}) = Q_{03}(t)\delta(\mathbf{r})$ is a function describing a point source. This source is located at the point R = 0 (where R, θ are spherical coordinates, z = R cos θ) and excites an intense shock wave which departs to infinity. Since the properties of the medium depend only on the single coordinate z, the source is a point, and its impulse is oriented along the z axis, the solution of Eq. (1) will have axial symmetry. It is known [5, 6] that in a wave moving upward the velocity of front displacement changes nonmonotonically, passing through a minimum at R = 1.5H/cos θ [6] (the analytical calculations of [5] give a

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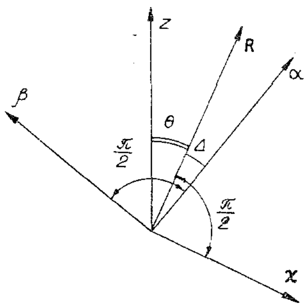

Fig. 1

coarse result: R = 4H/cos θ). In the present study we will analyze the evolution of an intense shock wave (Mach number M > 1) in that stage in which the shock front velocity u increases monotonically: $R \geqslant R_0 > 1.5H/\cos\theta$, $0 \leqslant \theta \leqslant \pi/2$. For intense shock waves it is permissible study the field on the front, without considering wave structure behind the front [9-11]. It is then possible to obtain approximate analytical expressions for the fields on the front.

Let $Q_{01}(t)$, $Q_{02}(t)$, $Q_{03}(t)$ be functions such that the source in Eq. (1) at a distance R = R_0 creates a shock wave with a profile in the form of a "discontinuity." We assume that the shock front velocity u(R_0, θ) is known. We require that on the front well-known relationships for a "discontinuity" be satisfied:

$$v = \frac{2a_0}{\gamma+1}\left(\mathrm{M} - \frac{1}{\mathrm{M}}\right), \quad p = \rho_0(t)\, a_0^2\left[\frac{2}{\gamma+1}\mathrm{M}^2 - \frac{\gamma-1}{\gamma+1}\right],$$
$$\rho = \rho_0(z)\frac{\gamma+1}{\gamma-1}\left(1 + \frac{2}{\gamma-1}\frac{1}{\mathrm{M}^2}\right)^{-1}, \quad a\rho = a_0\rho_0(z)\,\mathrm{M}\sqrt{\frac{2\gamma}{\gamma-1}}\,\mathrm{E}_1(\mathrm{M}), \tag{2}$$

where $\mathrm{M} = \frac{u(R,\theta)}{a_0} > 1$; $\mathrm{E}_1(\mathrm{M}) = \left[1 + \frac{4\gamma - (\gamma-1)^2}{2\gamma(\gamma-1)\mathrm{M}^2} - \frac{1}{\gamma \mathrm{M}^4}\right]^{1/2}\left[1 + \frac{2}{(\gamma-1)\mathrm{M}^2}\right]^{-1}$; a, a_0 are the speed of sound and the undisturbed value of that quantity; γ is the heat capacity ratio at constant pressure and volume.

As in [9], we introduce the nonlinear ray coordinates α, β, related to the shock front (Fig. 1). We assume that the reference vector $\mathbf{e}_\alpha$ is perpendicular to the shock front ($\mathbf{e}_\alpha \perp \mathbf{e}_\beta$), and moreover that $(\mathbf{e}_R, \mathbf{e}_\alpha) = \cos\Delta$. As length elements along the axes $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. We take respectively M dα and A dβ, A being a dimensionless function describing the cross section of the ray tube [9]. Thus, we use here coordinates with inhomogeneous nonlinear length scales. The shock wave can be described by three functions: $\Delta(\alpha, \beta)$, $\mathrm{M}(\alpha, \beta)$, and $A(\alpha, \beta)$, which satisfy two nonlinear "geometric dynamics" equations [9]:

$$\partial\Delta/\partial\beta = (1/\mathrm{M})\,\partial A/\partial\alpha; \tag{3}$$

$$\partial\Delta/\partial\alpha = (-1/A)\partial \mathrm{M}/\partial\beta. \tag{4}$$

In essence, Eqs. (3), (4) represent the condition of invariance of the square of the length element $\mathrm{M}^2 d^2\alpha + A^2 d\beta^2$ upon transition to the original coordinate system R, θ. We obtain the lacking third equation on the basis of the approximate "characteristic law" of [10, 11], assuming that the shock wave profile has the form of a discontinuity, Eq. (2), and identifying the discontinuity with the surface of the characteristics C_+ of the gas dynamics equations (1). We will use the approximation of a one-dimensional field propagating in a thin ray tube along the coordinate $\boldsymbol{\alpha}$. In such a ray tube with gas-impermeable walls when the conditions of weak dependence of the fields on coordinate β

$$\left|\frac{\partial}{\mathrm{M}\partial\alpha}(A\rho v_\alpha)\right| \gg \left|\frac{\partial}{A\partial\beta}(A\rho v_\beta)\right|, \quad \left|V_\alpha \frac{\partial}{\mathrm{M}\partial\alpha}\begin{pmatrix} v_\alpha \\ p \\ \rho \end{pmatrix}\right| \gg \left|v_\beta \frac{\partial}{A\partial\beta}\begin{pmatrix} v_\alpha \\ p \\ \rho \end{pmatrix}\right|$$

are satisfied, we have the approximate relationships

$$\operatorname{div}(\rho\mathbf{v}) \approx \frac{\partial}{A\mathrm{M}\,\partial\alpha}(A\rho v), \; v \approx v_\alpha, \; \frac{d}{dt} \approx \frac{d}{dt} + v\frac{\partial}{\mathrm{M}\,\partial\alpha}$$

TABLE 1

	Spherical wave									
	[6]	calculation by Eq. (10)		self-similar solution for A = A(R) [4]		parabolic approximation for A = A(R) [4]		[5]	
γ	$b_*(\gamma)$	$b(\gamma)$	$\delta(\gamma)$ %	$b(\gamma)$	$\delta(\gamma)$ %	$b(\gamma)$	$\delta(\gamma)$ %	$b(\gamma)$	$\delta(\gamma)$ %
1,1	0,117	0,1319	+12,7					0,04167	—64,4
1,2	0,146	0,1530	+4,79					0,07143	—51,1
1,4	0,174	0,1798	+3,33	0,1269	—27,1	0,1294	—25,6	0,1111	—36,1
1,5	0,184	0,1859	+1,03					0,1250	—32,1
5/3	0,195	0,1926	—1,23					0,1429	—26,7
2,0	0,211	0,2000	—5,21					0,1667	—21,0

and system (1) simplifies:

$$\frac{\partial\rho}{\partial t} + \frac{1}{A}\frac{\partial}{\mathrm{M}\,\partial\alpha}(A\rho v) \approx Q_1,$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\mathrm{M}\,\partial\alpha} + \frac{1}{\rho}\frac{\partial p}{\mathrm{M}\,\partial\alpha} + (\mathbf{g}, \mathbf{e}_\alpha) \simeq (\mathbf{Q}_2, \mathbf{e}_\alpha), \frac{\partial p}{\partial t} + v\frac{\partial p}{\partial\alpha} - a^2\left(\frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\mathrm{M}\,\partial\alpha}\right) \approx Q_3. \tag{5}$$

From system (5) for the characteristics C_+ at $0 \le \theta \le \pi/2$, $R > R_0$ we obtain

$$\frac{dp}{\mathrm{M}\,d\alpha} + \rho a\frac{dv}{\mathrm{M}\,d\alpha} = -\frac{a^2\rho v}{A(v+a)}\frac{dA}{\mathrm{M}\,d\alpha} - \frac{\rho a g\cos\theta}{v+a},\ \mathrm{M}\frac{d\alpha}{dt} = v + a. \tag{6}$$

The system of equations (2), (6) gives the desired third equation for finding Δ, M, A:

$$\frac{1}{\mathrm{E}_2}(\mathrm{M})\frac{d\mathrm{M}^2}{d\alpha} - \frac{1}{\rho_0}\frac{d\rho_0}{d\alpha}\left(2\mathrm{M}^2 - \frac{\gamma-1}{\gamma}\right) =$$
$$= \frac{(\gamma+1)}{\rho_0 a_0}\left\{\frac{a^2\rho v}{v+a}\right\}\frac{1}{A}\frac{dA}{d\alpha} - \frac{(\gamma+1)g}{\rho_0 a^2}\left\{\frac{\rho a\mathrm{M}}{v+a}\right\}\cos\theta,\ \mathrm{E}_2(\mathrm{M}) = \left[2\left(1+\frac{1}{\mathrm{M}^2}\right)\sqrt{\frac{2\gamma}{\gamma-1}}\,\mathrm{E}_1(\mathrm{M})\right]^{-1}. \tag{7}$$

In the stage $R > R_0$ we must deal with the homogeneous system of equations (3), (4), (7) and replace the solution of the problem with sources Q_1, $\mathbf{Q}_2$, Q_3 by a boundary problem. The functions within the curly brackets of Eq. (7) depend solely on M. The role of the sources Q_1, $\mathbf{Q}_2$, Q_3 appears as creation of a profile in the form of a discontinuity and specification of the shock front velocity $u(R_0, \theta)$ on a sphere of radius R_0. We solve system (3), (4), (7) approximately, assuming that the local form of the shock front differs only slightly from spherical. The functions M and A will then vary slowly with angular coordinate θ as compared to their change as functions of R. We transform to a system of local orthogonal coordinates R, χ (see Fig. 1), attached to a point on the shock wave fronts УВ: $\chi = x\cos\theta - z\sin\theta$, $R = x\sin\theta + z\cos\theta$, $\mathrm{M}d\alpha = dR\cos\Delta + d\chi\sin\Delta$, $Ad\beta = -d\chi\cos\Delta + dR\sin\Delta$ (the vectors **x**, **z**, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ lie in one plane). We replace system (3), (4) approximately by the equations

$$\frac{\partial}{\partial\chi}\left(\frac{\sin\Delta}{A}\right) + \frac{\partial}{\partial R}\left(\frac{\cos\Delta}{A}\right) = 0,\quad \frac{\partial}{\partial\chi}\left(\frac{\cos\Delta}{\mathrm{M}}\right) - \frac{\partial}{\partial R}\left(\frac{\sin\Delta}{\mathrm{M}}\right) = 0, \tag{8}$$

which have the solution $\Delta = \Delta(\chi)$, $A/\mathrm{M} = \mathrm{const}$, $A = A(R)$, $\mathrm{M} = \mathrm{M}(R)$.

Upon transition from Eqs. (3), (4) to Eq. (8) the angle θ takes on the role of a parameter, the ray coordinates become R, χ and as a consequence we obtain a condition for synchronous change of the functions A and M: $A/M \approx$ const. In the case to be considered further $M \gg 1$, Eq. (7) can be simplified to $\frac{\lambda_1(\gamma)}{\mathrm{M}}\frac{d\mathrm{M}}{d\alpha} + \frac{1}{A}\frac{dA}{d\alpha} + (1+\lambda_2(\gamma))\frac{1}{\rho_0}\frac{d\rho_0}{d\alpha} \approx 0$ or

$$A/A_0 = (\mathrm{M}/\mathrm{M}_0)^{-\lambda_1}\left[\frac{\rho_0(R\cos\theta)}{\rho_0(R_0\cos\theta)}\right]^{-(1+\lambda_2)}. \tag{9}$$

here $\lambda_1(\gamma) = 1 + 2/\gamma + \sqrt{2\gamma/(\gamma-1)}$; $\lambda_1(1.4) = 5.0743227$; $\lambda_2(\gamma) = \sqrt{(\gamma-1)/2\gamma} - (\gamma-1)/\gamma$; $\lambda_2(1{,}4) = 0.0922502$; A_0, M_0 are the values of the functions A, M at $R = R_0$. The system of Eqs. (8), (9) have the solution

$$\mathrm{M}(R)/\mathrm{M}_0 = A(R)/A_0 = \exp\left[(b(\gamma)/H)(R-R_0)\cos\theta\right],$$
$$\Delta(\chi) = -(\chi/H)b(\gamma)\cos\theta,\ b(\gamma) = (1+\lambda_2(\gamma))/(1+\lambda_1(\gamma)). \tag{10}$$

Plane wave							
self-similar solution [8]		self-similar solution [7]		self-similar solution [3]		$b(\gamma) = \left(2 + \sqrt{\frac{2\gamma}{\gamma - 1}}\right)^{-1}$ [10]	
$b(\gamma)$	$\delta(\gamma)$ %	$b(\gamma)$	$\delta(\gamma)$ %	$b(\gamma)$	$\delta(\gamma)$ %	$b(\gamma)$	$\delta(\gamma)$ %
0,126	+16,2					0,149	+27,3
0,155	+6,16	0,154	+6,16	0,1545	+5,82	0,183	+25,3
				0,1835	+5,46	0,215	+23,3
0,176	−3,26			0,193	+4,89	0,225	+22,0
0,204	+4,62	0,204	+4,62	0,204	+4,61	0,236	+21,0
0,219	+3,79			0,219	+4,10	0,250	+28,2

According to Eq. (10) the cross-sectional area of the ray tube A(R) varies exponentially. The hypothesis of an exponential dependence A(R) was used in [3, 4] to construct a self-similar solution of the second sort, but synchronous change of A(R) and M(R) was lacking. In [5] the authors commenced from the assumption $A(R) \sim R^2$. A comparison of Eq. (10) for the parameter $b(\gamma)$ with data of other studies is given in Table 1, where $b = b_*(\gamma)$ is the result of numerical integration of the gas dynamics equations [6]. The parameter $\delta(\gamma) = (b(\gamma) - b_*(\gamma))/b_*(\gamma)\cdot 100\%$ is the relative error of the approximate analytical results. Table 1 also presents data from [3, 7, 8, 10]. On a planar shock wave in an inhomogeneous atmosphere since those values are sometimes used to describe spherical waves. Equation (10) achieves highest accuracy at $\gamma \approx 1.5$, with the error comprising units of percent for $\gamma = 1.2$-2.0, while at $\gamma = 1.1$, $\delta = 12.7\%$. For any values γ the results of Eq. (10) are more accurate than those of [4, 5] for a spherical wave. For a plane wave at $\gamma = 2.0$ the result of [8, 3] is much closer to $b_*(\gamma)$ of a spherical wave from [6] than Eq. (10). Strictly speaking, comparison of data for plane and spherical waves is incorrect, and moreover the case $\gamma = 2.0$ corresponds to a frozen degree of ionization for low medium densities (which situation is of little interest for intense shock waves). We note that the accuracy of the method of [9-11] cannot be established in the general case. The closeness of the data of [6] and Eq. (10) is evidence of the advantages of the method of [9-11]. A significant factor in obtaining Eqs. (8)-(10) was the decrease in density of the undisturbed medium as the shock wave propagates. This ensures an increase in shock wave velocity with increase in R for $R > R_0$. The method of [9-11] cannot be used for $\pi/2 < \theta \le \pi$ and $H = \infty$ (in a homogeneous medium), since a decrease in shock wave velocity occurs with increase in R. In such a situation the wave changes from intense ($M > 1$) to weak ($M \to 1$).

We will consider the evolution of the intense shock wave ($M \gg 1$) of Eqs. (2), (10). In our approximation $M = M(R)$ and is independent of χ ($\chi \perp \mathbf{R}$). This means that we have the local property $\mathbf{u} \| \mathbf{R} \| \boldsymbol{\alpha}$ and from Eq. (10) with consideration of $u = dR/dt$ we obtain an expression to describe shock front evolution

$$\frac{b(\gamma)}{H}[R(t, \theta) - R_0] = -\frac{1}{\cos\theta}\ln\left[1 - \frac{t\cos\theta}{t_\infty(0)}\right] \quad (t_\infty(\theta) \equiv H(b(\gamma)a_0 M_0 \cos\theta)^{-1}). \tag{11}$$

At $t = t_\infty(\theta)$ there is a "break" of the shock front at infinity over a finite time, i.e., "explosive" instability is realized. Initially the "break" occurs in the direction $\theta = 0$ at the time $t = t_\infty(0)$. With increase in time the angular sector θ encompassed by this "break" expands. Figure 2 shows the form of the shock wave as a function of dimensionless time $\bar{t} = t/t_\infty(0)$ (lines 1-7 correspond to $\bar{t} = 0.5, 0.75, 0.95, 1.0, 1.1, 1.25, 1.5$) with the radius vector $\bar{r}(\theta)$ taken in the form of the dimensionless quantity $b(\gamma)\,[R(t, \theta) - R_0]H^{-1}$. The maximum horizontal distance traversed by the shock wave by the time $t = t_\infty(0)$ can be estimated from the expression $\max r = R_0 + 1.23H/b(\gamma)$ ($R > 1.5\ H$, $b^{-1}(\gamma) \approx 5.7$). This means that $\max r > 8.5H$.

The horizontal distance in the plane $\theta = \pi/2$ traversed by the shock wave by the time $t = t_\infty(0)$ $r(\theta = \pi/2) = R_0 + Hb^{-1} > 7.2\ H$. For comparison we can perform an estimate from the data of [1]: $\max r \approx 2.04H$. If we take $a_0 \approx 400$ m/sec, $H \approx 10^4$ m, $b^{-1} \approx 5.7$, we find a value of the parameter $t_\infty(0) \approx 2.4/M$ min ($M_0 > 1$).

In the region of departure of the shock front to infinity the assumption of weak dependence of the fields on angle θ becomes invalid together with the conditions for applicability of Eqs. (10), (11). From Eqs. (10), (11) we obtain an expression for evolution of the shock wave velocity

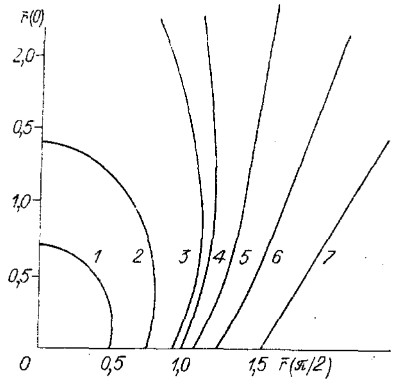

Fig. 2

$$u(t,\,\theta) = (H/b(\gamma))\,[t_\infty(0) - t\cos\theta]^{-1},\ \mathrm{M}(t,\,\theta) = \mathrm{M}_0 t_\infty(0)/(t_\infty(0) - t\cos\theta). \tag{12}$$

Considering Eqs. (2), (10), and (12) for M >> 1, we can obtain expressions for the v, p, ρ, a fields in the shock wave.

As an example we present expressions for the pressure:

$$p(R,\theta) = D_1(\theta)\exp\left[-(1-2b)\frac{R\cos\theta}{H},\quad p(t,\theta) = D_2(\theta)\left[1 - \frac{t\cos\theta}{t_\infty(0)}\right]^{\frac{1-2b}{b}},\right.$$

$$D_1(\theta) = \frac{1}{\gamma+1}2\mathrm{M}_0^2 a_0^2 \rho_0(0)\exp\left(-\frac{2bR_0}{H}\cos\theta\right),$$

$$D_2(\theta) = \frac{1}{\gamma+1}2\mathrm{M}_0^2 a_0^2 \rho_0(0)\exp\left(-\frac{R_0}{H}\cos\theta\right).$$

As $R \to \infty$ $p \to 0$ $(t \to t_\infty(\theta))$.

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